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FINAL REPORT ON AFOSR GRANT 77-3358

6/1/78 to 5/31/79

Charles L. Dolph

During the past year definitive answers were achieved for several questions raised in the proposal of February 1978. These include:

1. An examination of the relationship between the scattering matrix (the Fourier transform of the scattering operator) and the integral equations used in the Singularity Expansion Method (SEM) established that only the complex poles off the axis are intrinsically associated with the scatterer. While it has been known in special cases that those on the axis do not contribute to the field (e.g. Dolph, "The integral equation method in scattering theory", Problems in Analysis, Ed. R. C. Gunning, Princeton University Press, 1970. An AFOSR Symposium), this appears to be the first time this relationship has been clearly exhibited. Since the scattering matrix can be shown to be analytic in a half-plane containing the axis, any integral equation should exhibit the same properties for this region. Those of SEM fail to do so but other integral equations which do can be given in all cases.

This result raises the interesting and as yet open question as to whether the poles of the integral equation of SEM in addition to yielding poles not intrinsic to the body might yield only a subset of the complex poles of the scattering matrix. No examples where this occurs are yet known but there is no proof that it cannot happen. In contrast, if an integral equation using the Green's function in contrast to the free space Green's function is used, A. G.

The Singularity Expansion Method and Complex Singularities of Exterior Scalar and Vector Scattering in Acoustics and Electromagnetic Theory

FINAL REPORT ON AFOSR GRANT 77-3358

6/1/78 to 5/31/79

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Principal Investigator: Charles L. Dolph

Table of Contents

Ramm has recently shown that there is a one-to-one correspondence between the poles of this Green's function and the zeros of the eigenvalues as they are used in the Eigenmode Expansion theory. This will be elaborated on in future work.

2. The relationship between the eigenvalues of the integral equations of SEM and the complex eigenvalues of the associated partial differential equations -- whether scalar or vector. In particular, the integral equations of SEM have at most two eigenvalues $\pm$ and these are functions of the at-most-denumerable number of complex eigenvalues of the associated differential equations.

3. These results were first presented in an invited address at a special session on Integral Equations at the annual meeting of the American Mathematical Society held in Biloxi, Mississippi, January 24-28, 1979. An abstract of this talk is attached.

Moreover, because of the unfamiliarity of many SEM workers with the mathematics involved, several attempts were made to write an account suitable for the IEEE Transactions on Antennas and Propagation. C. E. Baum and others felt that the first two drafts were too detailed and technical, so that a third attempt, with many simplifications, was prepared. The resulting paper, joint with S. K. Cho of the Radiation Laboratory of the University of Michigan, has been submitted to the above journal. It is entitled "On the relationship between the singularity expansion method and the mathematical theory of scattering".

While a copy of this paper in its submitted form is attached it is doubtful that the paper will appear exactly as here. Correspondence with C. E. Baum and one of the referees, Professor Wilson Pierson of the University of

Kentucky seem to indicate that certain sections will still have to be expanded. Knowledge of this paper's existence seems quite wide spread in view of the number of requests that have been received for preprints from various SEM workers.

4. As indicated in the 1979 proposal it was felt that the theory of non-self-adjoint operators in Hilbert space would have significant implications for that part of the electrical engineering formalism known as the Eigenmode Expansion Method (EEM) particularly in relevance to the use of this formalism in developing equivalent circuits.

A survey of the Russian literature revealed that a significant start in this direction had been made in both the scalar and vector cases by several Russian workers. In particular, papers by A. G. Ramm and M. S. Agranovic in the scalar case, made it evident a forteriori that the EEM formalism based on the impedance integral equation would not be correct in general and thus the work on equivalent circuits would have to be critically examined.

In this connection C. L. Dolph, with considerable help from V. Komkov, reviewed the book: The Generalized Methods of Eigenvibrations in the Theory of Diffraction byVoitovic, Kacenelenbaum and Sivov, as well as the mathematical appendix Spectral Properties of Diffraction Problems by Agranovic. An unedited copy of this review is attached but it is expected that the final version will appear in the September or October issue of Mathematical Reviews for this year.

Because of its importance two attempts at translation are underway. A machine translation, arranged by Dr. R. Buchal of AFOSR, and an American Mathematical Society translation, arranged by the author and the editors of Mathematical Reviews.

As the attached abstract indicates, much of the above material was presented in a talk by C. L. Dolph at the International Symposium on Recent Developments in Classical Wave Scattering held at Ohio State University June 25-27, 1979.

A manuscript by C. L. Dolph, V. Komkov and R. A. Scott entitled "A Critique of the Singularity Expansion and Eigenmode Expansion Method" will appear in book form in the conference proceedings to be published by Pergamon Press. A copy of this manuscript is attached. As this paper contains the relevant English translations of the Russian literature, copies of its bibliography were also distributed (see attached copy). Additional copies of this bibliography will be furnished on request.

5. With the aid of Dr. R. Buchal of AFOSR and Professor F. Gehring and Lee Zukowski of the Mathematics Department of the University of Michigan and many others, including the author, A. G. Ramm, formerly of the University of Leningrad, was successfully brought to the University of Michigan. Since his arrival on June 5th, 1979, he and the authors have been actively planning and beginning to implement further research in the areas under discussion. A detailed plan will be available shortly. A list of publications and curriculum vitae of A. G. Ramm is attached.

Abstract of Talk at American Mathematical Society Meeting
Biloxi, Mississippi, January 24-27, 1979

C.L. Dolph, University of Michigan, Ann Arbor, Michigan 48104. Fredholm integral equations, scattering theory, and the singularity expansion method (SEM).

Since 1971 SEM has been extensively used in electromagnetic theory e
vident from the review article by C.E. Baum: Emerging technology for trans
d broad band analysis and synthesis of antennas and scatterers," Proc. I.
64, 1976, pp. 1598-1616. The Fredholm integral equations and their complex
ngularities used in SEM will be discussed and SEM will be interpreted in
rms of mathematical scattering theory.

ON THE RELATIONSHIP BETWEEN THE SINGULARITY EXPANSION METHOD AND THE MATHEMATICAL THEORY OF SCATTERING

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1. Introduction

The singularity expansion method (SEM) and the mathematical theory of scattering have been extensively developed independently of each other during the last decade or so. To our knowledge the relationship between them has not been discussed. In this paper the implications for SEM of the mathematical scattering theory will be explored. As a result some important conclusions of SEM will be seen to also follow from scattering theory and, at the same time, certain gaps between the two approaches will appear.

The SEM was introduced by C. Baum in 1971 [1] as a technique for solving transient electromagnetic scattering from bodies of finite extent. Since then, as reference to [2], [3] and [4] indicate, not only has it been widely used, but an extensive formalism has been developed for it. Succinctly, the SEM is a generalization of well-known technique of linear circuit theory in which singularities of a transfer function in the complex frequency plane are used to determine the transient response by the Heaviside expansion theorem. More details of the procedures of the SEM will be given in Section 2.

Mathematical scattering theory as used here originates in the work of Lax and Phillips [13, 14, 15] for the scalar case and its generalization by Schmidt [23, 24] for the electromagnetic case. An independent development of the theory based upon integral equations similar to those employed in the SEM theory is due to Shenk and Thoe [25], [26], and [27] in the scalar case and to Pyz'janov [19], [20], and [21] in the electromagnetic case.

The relevant parts of this theory will be sketched in Section 3 as well as its implications for SEM. In Section 4 additional aspects of the SEM formalism will be commented upon and some questions raised as to its interpretation.

2. The SEM Formalism and a Mathematical Interpretation

In view of several descriptions of the SEM given in the reference already quoted, only a very brief sketch of the relevant formalism will be given here. The discussion will further be limited to the electromagnetic case for convex perfectly conducting bodies.

Maxwell's equations are first subjected to a two-sided Laplace transformation with respect to the time and the initial value problem replaced by a time-independent boundary value problem for a reduced wave equation for the electric or magnetic field subject to the corresponding boundary condition for perfect conductivity and a vector radiation condition. A representation for the scattered field is then sought in the form of an integral representation. Application of the boundary condition yields an integral equation (or in some instances an integral-differential equation) whose kernel is given in terms of some free space Green's function or dyadic. The unknown in the integral equation is the response (often a current density on the scatterer) induced by the incident wave. The so-called natural frequencies $\{s_n\}$ are the values of s for which the corresponding homogeneous integral equation has non-trivial solutions. At such a value of s, the inverse to the integral equation will not exist and the values of s will be poles of the inverse. The resulting residue series which leads to coupling coefficients will not be discussed here. In actual practice the homogeneous integral equation is replaced by a matrix equation by some finite approximation technique such as that of the method of moments. The poles are then sought approximately as the zeros of the determinant of this matrix.

The formal procedure, leaving aside the relation between the solutions of the integral equation and the matrix equation, has been justified mathematically in only a few cases. The most complete treatment is due to Marin [17]. For a perfectly conducting body he used the Franz H-field representation and treated the resulting magnetic field integral equation (MFIE) for the induced current density; namely,

$$(I - 2\bar{L})\cdot\underline{J} = \underline{J} - 2\int_\Gamma \underline{n} \times (\nabla G \times \underline{J})\,d\sigma = 2(\underline{n} \times \underline{H}^{inc}).$$

Here $\underline{n}$ is the outward bound unit normal to the scatterer and G is scalar free space Green function

$$G = \frac{e^{-s|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|} \quad .$$

Solutions are sought in the Hilbert space of elements $\underline{J}$ that are tangent to Γ and square integrable on Γ where Γ is assumed sufficiently smooth. Specifically the integral equation

$$(I - 4\bar{L}^2)\cdot \underline{J} = 2(I + 2\bar{L})\cdot(\underline{n} \times \underline{H}^{inc})$$

is shown to be Fredholm and treated by the method of Carleman. The operators $(I - 2\bar{L})$ and $(I + 2\bar{L})$ are shown to have inverses for the same values of s and consequently these inverses exist whenever this last integral equation has an inverse. From this it follows that $(I - 2\bar{L})^{-1}$ has a pole at $s = s_n$ whenever the homogeneous integral equation $(I - 2\bar{L})\cdot\underline{j} = 0$ has a non-trivial solution at $s = s_n$.

The poles so determined lie in the union of two sets. One set consists of poles corresponding to interior resonances. These are purely imaginary and their occurrence is due to the method. They do not contribute to the scattered field and hence are not intrinsic to the scattering problem. The other set consists of poles corresponding to exterior resonances and they are not purely imaginary. They are in fact eigenvalues of the vector wave equation

$$\nabla \times \nabla \times \underline{H}_n + \frac{s_n^2}{c^2}\,\underline{H}_n = 0$$

for the exterior scattering problem.

The claim that these poles (exterior resonances) are intrinsic to the scattering problem will be further substantiated by their interpretation as poles of the scattering matrix in Section 3.

Marin's conclusion can also be deduced directly from the analytic Fredholm theorem. In its general form the theorem is due to Steinberg [28] who established it for a general Banach space.

Recall that a Banach space B is a linear space in which a "distance" is defined by a norm, usually denoted by $\|\ \|_B$ and it is complete in the sense that every Cauchy sequence has a limit (in norm) in the space. In general there is no inner product. Let $\mathcal{L}(B)$ be the set of bounded operators on B A norm in $\mathcal{L}(B)$ can be defined for a linear operator A by

$$\|A\|_{\mathcal{L}} = \sup_{x \in B} \frac{\|Ax\|}{\|x\|} .$$

An operator A in $\mathcal{L}(B)$ is called compact (or completely continuous) if and only if for every bounded sequence $\{x_n\}$ in B , $\{Tx_n\}$ has a convergent subsequence in B .

A family of operators $T(s)$ depending upon s is called an analytic family in the neighborhood of s_0, if

$$T(s) = \sum_0^{\infty} T_n (s-s_0)^n ,$$

where convergence is in the operator norm and where T_n is in $\mathcal{L}(B)$.

The theorem can be stated as follows:

If $T(s)$ is an analytic family of compact operators for $s \in K$, an open connected subset of the complex plane, then either $(I - T(s))$ is nowhere invertible in K or $((I - T(s))^{-1}$ is meromorphic in K . At a pole, the equation $\psi - T(s_0)\psi = 0$ has a non-zero solution in B .

If B is a separable Hilbert space the inner product can be used to express the residue in terms of operators of finite rank. That is, for each pole there will exist an N and two sets of linearly independent functions $\phi_1,\ldots,\phi_N$, $\psi_1\ldots\psi_N$ such that the corresponding residue can be expressed as

$$\sum_1^N (\phi_k,\ldots)\psi_k$$

where (f,g) has been used for the inner product. A proof of this can be found in Reed and Simon [22].

It may be shown directly, or Marin's argument can be used to conclude, that the MFIE is a Fredholm integral equation of the second kind having an integral operator which is compact. It is also not difficult to verify that the MFIE yields a family of operators analytic in s . Marin's conclusion then follows from the Hilbert space analytic Fredholm theorem.

However, since in most SEM applications it is necessary to resort to numerical methods, it is more appropriate to seek solutions in the space of continuous functions. Such a space here would consist of the Banach space of all continuous $\underline{J}$ tangent to Γ when the norm is taken to be

$$\|\underline{J}\|_c = \max_{x\in\Gamma} |\underline{J}(x)| .$$

The Fredhom operator of the MFIE will still be a compact analytic family in s and consequently the same conclusions about the natural frequencies will follow.

Unlike the papers by Marin other SEM papers are difficult to interpret mathematically since neither the properties of the integral equations nor the spaces in which solutions are sought are specified. Some papers also employ integral equations of the first kind which, it is claimed, lead to approximate solutions which are easier to handle numerically. It should, however, be recognized that integral equations of the first kind cannot, unless of certain special types like Wiener-Hopf, be used to establish existence and uniqueness properties. Integral equations of the first kind, in whatever space their solutions are sought, are not a well set problem in the sense of Hadamard so that some method of regularization such as one of the methods of Tihonov [30] should be used. While such methods have been used in other electromagnetic problems [cf. (8)] they do not appear to have occured in connection with SEM. It should also be emphasized that for scatterers of finite extent, the Franz E-field representation will not lead to an integral equation, no matter how smooth the boundary fields are assumed to be.

In view of this the subsequent interpretation in terms of scattering theory will be limited to those problems involving Fredholm equations of the second kind.

3. The scattering operator and scattering matrix.

Several considerations have influenced our decision to restrict further discussion to scalar rather than vector wave equations. Not only are all the essential ideas present in this simpler case but all proofs of the assertions we will make are readily available. This is, unfortunately, not true in the other case. For example the theorems about the scattering operator and the representation of the scattering matrix are stated without proof by Schmidt [23,24]. He indicates that they follow directly from the corresponding results contained in the book by Lax-Phillips [13]. And while this is true, we do not expect our readers to have the necessary familiarity with the Lax-Phillips theory to find this easy to do. A reference to Beale's unpublished thesis [5] , where many of these details are carried out, will soon convince one of the difficulties involved. Further, as noted in the reviews of the papers by Pyz'janov [19], [20], [21], many open questions concerning the spaces and operators used by this author are also not specified and thus these papers are open to the same criticism as most of the papers in SEM.

Fortunately a straight forward extension of the Shenk-Thoe theory could be carried out for $s \neq 0$. This would involve replacing the representation for the scalar field developed by Werner [31] and used by them by Werner's subsequent representation of scattered electric field as given in [32]. The Shenk-Thoe procedure could then be followed step-by-step and the conclusions reached by Schmidt obtained in this way just as the conclusions of the Lax-Phillips theory can be reached by the Shenk-Thoe method in the scalar case.

While Laplace transformations in time have so far been used, henceforth they will be replaced by an equivalent Fourier transform.

The time convention used will be that of most papers in physics and under it, the upper half plane replaces the right-hand half plane as the domain of analyticity. This is also the convention used by Shenk-Thoe and its use allows one to state spectral properties in the usual form. It amounts to introducing $k = is$. It is not that of Lax-Phillips and Schmidt where "i" would be "-i". Thus outgoing waves here would be incoming in their sense and vice-versa.

Let Ω be an unbounded free space $(\Omega \subset E^n,\ n = 2,3)$ exterior to a sufficiently smooth convex body with the surface Γ. The time-dependent acoustic scattering problem seeks the solution for the wave equation for the scattered field u^s

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u^s(\bar{r},t) = 0 \ , \ \bar{r} \in \Omega \ , \ t > 0$$

subject to the initial conditions

$$\left.\begin{aligned} u^s(\bar{r},0) &= f_1(\bar{r}) \ , \\ \frac{\partial}{\partial t} u^s(\bar{r},0) &= f_2(\bar{r}) \ . \end{aligned}\right\} \quad t = 0 \ , \ \bar{r} \in \Omega \qquad (3.1)$$

and one of the following three boundary conditions for $t \geq 0$:

Dirichlet: $\quad u^s(\bar{r}_0,t) = g_1(\bar{r}_0,t)$,

Neumann: $\quad \dfrac{\partial}{\partial n_0} u^s(\bar{r}_0,t) = g_2(\bar{r}_0,t), \quad \bar{r}_0 \in \Gamma$

Robin: $\quad \left(\dfrac{\partial}{\partial n_0} + \alpha\right) u^s(\bar{r}_0,t) = g_3(\bar{r}_0,t)$, where $\frac{\partial}{\partial n_0} = \bar{n}_0 \cdot \nabla_0$, $\bar{n}_0$ denoting the unit normal vector at the observation point on Γ directed into Ω.

In the scattering theory, one views these problems as perturbations of the free space, or unperturbed problem (without the scattering obstacle) such that, for large time $|t|$, the perturbed (or scattered) solutions become asymptotically equal to the free space solutions. The scattering operators which are thought to contain all the observable information about the nature of the scatterers represent measures of these perturbations. Let $U(t)$ denote operators which relate the states at $t = 0$ to those at $t > 0$ in the scattering problems. Similarly, let $U_0(t)$ denote the corresponding operators in the free space problems. If $f = (f_1, f_2)$ denotes the initial data, then $U(t)f$ solves a scattering problem in the sense that it represents the state at time $t > 0$, evolved from the initial state. Since the scattered solution is assumed to be asymptotically equal in large $|t|$ to the free space solution, one expects that there exist initial states, $f_0 \pm$, in the free space unperturbed problem such that $U_0(t)\, f_0 \pm$ are asymptotically indistinguishable from $U(t)f$ as $t \to \pm\infty$. This is made precise by introducing an appropriate Hilbert space with the energy norm

$$\|f\| = \int_\Omega [|f_1|^2 + |f_2|^2]\, dv$$

where f_1 and f_2 are compactly supported. For this space, the operators $U(t), U_0(t)$ are one-parameter groups and unitary. The asymptotic behaviors stated above are now expressed as

$$\|U(t)f - U_0(t)\, f_0- \| \to 0 \ , \ \text{as} \quad t \to -\infty \ ,$$

$$\|U(t)f - U_0(t)f_0+ \| \to 0 \ , \ \text{as} \quad t \to +\infty \ .$$

Since $U(t)$, $U_0(t)$ are unitary and form a group, the above expressions may be written as

$$\|U(-t)\ [U(t)f - U_0(t)\ f_0\pm]\| \to 0, \ \text{as} \quad t \to \pm\infty \ ;$$

or

$$\|f - U(-t)U_0(t)f_0\pm\| \to 0 \ , \ \text{as} \quad t \to \pm\infty \ .$$

The operators

$$W_\pm(t) = \lim_{t \to \pm\infty} U(-t)U_0(t) \ ,$$

exist, and are called the wave operators as is well known in the physics. $W_+(t)$, for instance, compares the asymptotic free space solution $U_0(t)f_0-$ with the asymptotic scattered solution $U(t)\ f_0+$. See the Figure 1 below.

Figure 1.

$$\begin{tikzcd}
f_0- \arrow[d, "\mathcal{S}"'] \arrow[dr, "W_-"] & \\
f_0+ \arrow[r, "W_+"'] & f
\end{tikzcd}$$

The map

$$\mathcal{S} : \ f_0- \longmapsto f_0+ \tag{3.2}$$

Figure 1.

$$\begin{tikzcd} f_0^- \arrow[dr, "W_-"] \arrow[dd, "\delta"'] & \\ & f \\ f_0^+ \arrow[ur, "W_+"'] & \end{tikzcd}$$

is called the scattering operator. That such operators $\mathcal{S}$ exist and are unitary in the Hilbert space can be established. As evident from Figure 1,

$$\mathcal{S} = W_+^{-1} W_- \quad .$$

Let $\mathfrak{F}$ denote the n-dimensional $(n = 2,3)$ Fourier transformation defined by

$$\mathfrak{F}(f) \equiv \tilde{f}(\bar{k}) = (2\pi)^{-n/2} \int_{-\infty}^{\infty} f(\bar{r})\, e^{-i\bar{k}\cdot\bar{r}}\, d^n\bar{r} \; . \tag{3.3}$$

The scattering matrix is then defined by

$$S = \mathfrak{F}\, \mathcal{S}\, \mathfrak{F}^{-1} \quad . \tag{3.4}$$

Next, consider an exterior homogeneous boundary value problem

$$\begin{aligned} (\Delta + k^2)\, \phi_{\pm}(\bar{r}) &= 0 \;, \quad \bar{r} \in \Omega \;, \\ \gamma[\phi_{\pm}(\bar{r}_0)] &= 0 \;, \qquad \bar{r}_0 \in \Gamma \;. \end{aligned} \tag{3.5}$$

Here γ is written for the Dirichlet, Neumann or Robin boundary condition.

Then,

$$\phi_{\pm}(\bar{r}) = e^{i\bar{k}\cdot\bar{r}} + v_{\pm}(\bar{r},\bar{k}) \tag{3.6}$$

are evidently the solutions of the problem. Although they are not in $L_2(\Omega)$, $\phi_{\pm}(\bar{r})$ are improper eigensolutions. $v_{\pm}(\bar{r},\bar{k})$ are the solutions of the boundary value problem

$$(\Delta + k^2)\, v_{\pm}(\bar{r},\bar{k}) = 0,\ \bar{r} \in \Omega .$$

$$\gamma[v_{\pm}\,(r_0,\bar{k}) \quad + \quad e^{i\bar{k}\cdot\bar{r}_0}] = 0,\ \bar{r}_0 \in \Gamma . \tag{3.7}$$

v_+ (v_-) is called an outgoing (incoming) diffracted plane wave; $\phi_{\pm}(\bar{r})$ are called distorted plane waves.

If a generalized transformation is defined by

$$\mathcal{D}_{\pm}(f) = \lim_{\rho\to\infty} (2\pi)^{-n/2} \int_{\Omega_\rho} f(\bar{r})\ \phi_{\pm}(\bar{r},\bar{k})^* d^n\bar{r}\ , \tag{3.8}$$

where $\Omega_\rho = \{\bar{r} \in \Omega\ :\ |\bar{r}| < \rho\}$ for some sufficiently large ρ and where $*$ denotes complex conjugate, then the scattering matrix S takes a simple form

$$S = \mathcal{D}_+ \mathcal{D}_-^{-1}\quad ,$$

which is equivalent to

$$S\mathcal{D}_-\ f = \mathcal{D}_+\ f\ .$$

Th[illegible] the scattering matrix maps $\phi_-(\bar{r},\bar{k})$ into $\phi_+(\bar{r},\bar{k})$. This form is particularly convenient for time-independent scattering theory. Since k will be continued to the complex plane, it is necessary to replace the usual radiation condition by a representation based on Green's formula. For $v_{\pm}(\bar{r},\bar{k})$ this takes the following form: for outgoing waves associated with the plus sign and incoming waves associated with the minus sign,

$$v_{\pm}(\bar{r},\bar{k}) = \int_{\Gamma} [v_{\pm}(\bar{r}',\bar{k}) \frac{\partial}{\partial n'} G^{\pm}(kR) - G^{\pm}(kR) \frac{\partial}{\partial n'} v_{\pm}(\bar{r}',\bar{k})] dA' , \quad (3.9)$$

where $G^{\pm}(kR) = \pm \frac{i}{4} (\frac{k}{2\pi R})^{(n-2)/2} H^{(1)\,(2)}_{(n-2)/2}(kR)$, $n = 2,3$

$R = |\bar{r}-\bar{r}'|$, $\bar{r} \in \Omega$, $\bar{r}' \in \Gamma$.

This representation implies that the corresponding integral of the right-hand side taken over a large sphere tends to zero as the radius of this sphere tends to infinity. It can be shown that this in turn is equivalent to the usual radiation conditions for $k \neq 0$ and $0 < \arg k \leq \pi$.

For $\bar{k}$ real, $r = |\bar{r}|$, $\bar{\theta} = \bar{r}/r$, $\bar{\omega} = \bar{k}/k$, $k = |\bar{k}|$, and for $|\bar{r}| \gg |\bar{r}'|$ on obtains the asymptotic form

$$v_{\pm}(\bar{r},\bar{k}) = v_{\pm}(r\bar{\theta},k\bar{\omega}) = \frac{e^{\pm ikr}}{r^{(n-1)/2}} [s_{\pm}(\bar{\theta},k,\bar{\omega}) + 0(1)] . \quad (3.10)$$

The functions $s_{\pm}$ in this last formula are called radiation patterns or transmission coefficients. In terms of them the scattering matrix takes the form

$$S(k)h(\bar{\theta}) = h(\bar{\theta}) + (ik/2\pi)^{(n-1)/2} \int_{|\bar{\omega}|=1} h(\bar{\omega}) s_{-}(-\bar{\theta},k,\bar{\omega})^{*} dS_{\bar{\omega}} \quad (3.11)$$

for $h(\bar{\theta}) \in L_2(S^{n-1})$, S^{n-1} being a surface of the unit ball in n-dimensional space. The above integral operator is compact for $k > 0$.

This representation may be derived from the scattering operator [26] or it may be verified by first postulating it and then invoking some form of the radiation condition and Rellich's uniqueness theorem, [13, 15].

The representation formulas for outgoing (incoming) waves $v_\pm$ imply that

$$s_\pm(\bar\theta,k,\bar\omega) = \pm i\,\frac{\beta(k)}{2\pi}\int_\Gamma [v_\pm(\bar y,k,\bar\omega)\,\frac{\partial}{\partial n'}\,e^{\mp ik\,\bar y\cdot\theta} - e^{\mp ik\,\bar y\cdot\theta}\,\frac{\partial}{\partial n'}\,v_\pm(\bar y,k,\bar\omega)]\,d\sigma_y \tag{3.12}$$

where $\beta(k) = (\mp\frac{ik}{2\pi})^{(\frac{n-1}{2})}$, $n = 2,3,\ k > 0$.

This relation establishes a one-to-one map between the analytical properties of $s_\pm$ and of $v_\pm$. Since it can be shown that $s_-(-\bar\theta,k,\bar\omega)^* = s_+(-\bar\theta,k^*,-\bar\omega)$ it follows that if the solution v_+ can be constructed and shown to have a meromorphic continuation from the positive real axis to the entire complex plane if n is odd and to the entire logarithmic Riemann surface $\{k \neq 0\ ,\ -\infty < \arg k < \infty$ if n is even , it will follow that the same is true for the scattering matrix. It also follows from the Shenk-Thoe theory that the scattering matrix is unitary for each positive k .

Shenk and Thoe construct the solution for v_+ for a more general problem in which surface Γ may have several components and admit, a perhaps empty, decomposition as the union of Γ_1 and Γ_2. On Γ_1 Robin boundary conditions are imposed ($\alpha = 0$ is allowed) while Dirichlet boundary conditions are imposed on Γ_2 . The scattered field is sought, following Werner [31] , as a superposition of a single layer, a double layer and a volume potential and the resulting system of Fredholm integral equations of the second kind is shown to be uniquely solvable for $\operatorname{Im} k \geq 0$ in an appropriately constructed Banach space. Steinberg's theorem is then invoked to establish the analytic properties of the scattering matrix. This method, while quite general, would not be useful for numerical computation because of the presence of the volume potential and, to our knowledge, it has never been employed for this purpose.

In contrast, for the Dirichlet and Neumann problem alternate representations using a complex combination of single and double layer potentials not only can be used to establish a 1-1 map , but is more useful for numerical calculations.

The representations are due to Brakhage and Werner [7] and to Leis [16] and take the form:

$$v_+(\bar{r},\bar{k}) = \int_\Gamma \left[\frac{\partial G^+(kR)}{\partial n_y} \mp i\tau G^+(kR)\right] \psi(\bar{y})d\sigma_y , \tag{3.13a}$$

$$v_+(\bar{r},\bar{k}) = \int_\Gamma \left[G^+(kR) \pm i\tau \frac{\partial G^+(kR)}{\partial n_y}\right] \psi(\bar{y})d\sigma_y . \tag{3.13b}$$

Here, $\tau = 1$ for $\operatorname{Re} k \geq 0$ and $\tau = -1$ for $\operatorname{Re} k < 0$. The resulting integral equations for a plane wave incidence are

$$\psi(\bar{x}) + 2 \int_\Gamma \left(\frac{\partial G^+}{\partial n_y} \mp i\tau G^+\right) \psi(\bar{y})d\sigma_y = -2e^{-i\bar{k}\cdot\bar{x}} \tag{3.14a}$$

and

$$\psi(\bar{x}) - 2 \int_\Gamma \left[\frac{\partial G^+}{\partial n_x} \pm i\tau \frac{\partial^2 G^+}{\partial n_x \partial n_y}\right] \psi(\bar{y})d\sigma_y = 2\frac{\partial}{\partial n_x} e^{-i\bar{k}\cdot\bar{x}} \tag{3.14b}$$

These equations have analytic compact integral operators in the Banach space of continuous functions and possess a unique solution for Im $k \geq 0$ and are meromorphic elsewhere. They therefore also provide a 1-1 map between the transmission coefficient, and the solutions to the reduced wave equation. It should be noted, however, that because of the higher order singularity in the Neumann problem it is necessary to use a method of regularization to obtain these results. This is carried out, following the method of Leis [16], in detail in Kussmaul [12] where the cylinder problem is treated numerically. The corresponding numerical treatment for the cylinder Dirichlet problem is given by Greenspan and Werner [10].

If SEM were specialized to this case the corresponding assumptions would be that

$$v_+ = \int_\Gamma \frac{\partial}{\partial n_y} G^+ \delta(y)\, d\sigma_y \tag{3.15}$$

for the Dirichlet problem and

$$v_+ = \int_\Gamma G^+ \mu(y)\, d\sigma_y \tag{3.16}$$

for the Neumann problem. The corresponding integral equation would be, respectively,

$$\delta(\bar{x}) + 2 \int_\Gamma \frac{\partial G^+}{\partial n_y} \delta(y)\, d\sigma_y = -2\, e^{-i\bar{k}\cdot\bar{x}} \tag{3.17}$$

$$\mu(\bar{x}) - 2 \int_\Gamma \frac{\partial}{\partial n_x} G^+ \mu(y)\, d\sigma_y = 2 \frac{\partial}{\partial n_x} e^{-i\bar{k}\cdot\bar{x}} \tag{3.18}$$

Since as shown in Kupradze [11] these representations do not provide a unique solution to the corresponding boundary value problem for the reduced wave equation, the non-trivial solutions at $\{k_n\}$ of the homogeneous equation are not in 1-1 correspondence with the poles of scattering matrix. The scattering matrix has no poles in the upper half-plane including the real axis and thus does not contain the k_n corresponding to interior resonances. Nor do these occur in the expression for the scattered field. A direct proof of this last fact for the Dirichlet problem can be found in Dolph [9]. The complex k_n, as may be seen by direct computation, to correspond to outgoing solutions of the reduced wave equation are complex eigenvalues of the exterior scattering problem and consequently correspond to poles of the s-matrix. These are method independent and intrinsic to the scattering body in contrast to the interior resonances which arise only because of the method.

Because these presentations fail to furnish a 1-1 map, it is also conceivable that there may be other poles of the scattering matrix not given by solutions of the homogeneous integral equations used in SEM. There are no known examples in which this occurs, but the problem must be considered open.

More explicitly the complex poles of the scattering matrix are in 1-1 correspondence with the solutions of the boundary value problem

$$\begin{aligned} (\Delta + k_n^2) w_n(\bar{r}, k_n) &= 0, \quad r \in \Omega \\ \gamma w_n(\bar{r}_0, k_n) &= 0, \quad r_0 \in \Gamma . \end{aligned} \tag{3.19}$$

The complex poles k_n, $n = 1,2,3,\ldots$,

$$k_n = (k_0)_n - i\, k'_n \ , \ k'_n > 0 \ ,$$

$$0 > \operatorname{Im}(k_1) > \operatorname{Im}(k_2) > \operatorname{Im}(k_3) > \ldots$$

lead to the representation of the solution of (3.1) in the form

$$u^s(\bar{r},t) = \sum_{n=1}^{\infty} \alpha_n e^{-ik_n t} w_n(\bar{r}) . \tag{3.20}$$

While the functions $w_n(\bar{r})$ grow exponentially, in odd dimensions the causality implies that solution are zero for time $t < \frac{r}{c}$, c being the speed of the wave propagation; after this time, the solution decays in time.

Lax-Phillips [13] proved that, under the Dirichlet boundary condition for $n = 3$, the scattering operator uniquely determines the scatterer. There are few theorems about the locations of the complex poles of the scattering except for those located on the negative imaginary axis (in the SEM, on the negative real axis) of the complex wave-number plane. Thus, Lax-Phillips showed that, under Dirichlet, or Neumann boundary condition in 3-dimension, if $\sigma = \operatorname{Im}(k)$, $\operatorname{Re}(k) = 0$ and if the scatterer contains a sphere of radius R_1, and is contained in a sphere of radius R_2, then the number of poles $N(\sigma)$ is

$$\liminf_{\sigma\to\infty} \frac{N(\sigma)}{\sigma^2} \geq \frac{1}{2}\left(\frac{R_1}{\gamma_0}\right)^2 ,$$

while, if the scatterer is star-shaped,

$$\limsup_{\sigma \to \infty} \frac{N(\sigma)}{\sigma^2} \leq \frac{1}{2}\left(\frac{R_2}{\gamma_0}\right)^2 ,$$

$$\gamma_0 = 0.66272 .$$

J. Beale [6] extended the above results to all n, i.e., $n = 2,3$, including the case of Robin condition. In particular, for i he showed that the scattering matrix has at most a finite number of poles on the purely imaginary (purely real) axis when a Fourier (Laplace) transformation is used. Furthermore, for $n = 2$, if the boundary condition is either Dirichlet, or Neumann, then there are no poles on the axis. The poles on the axis are, of course, purely decaying modes. The complex poles of the scattering matrix which are also the complex eigenvalues of $-\Delta$ in an exterior homogeneous boundary value problem for the Helmholtz equation are associated with the physical nature of the scattering.

Finally, we will quote an example due to Shenk-Thoe [27] to show that the complex poles of the scattering matrix are precisely those exhibited in the separation of variable solution.

Let S^{n-1}, $n = 2,3$, denote the unit ball in E^n. Then, decompose $L_2(S^{n-1})$ into finite dimensional subspaces H_ℓ of spherical harmonic of degree ℓ. Each of these spaces is an eigenspace of the operator $S(k)$, the scattering matrix, and its eigenvalue is

$$-\frac{H^{(2)}_{p+\ell}(ka)}{H^{(1)}_{p+\ell}(ka)} , \; p = \frac{n-2}{2} , \; k > 0$$

where a denotes the radius of sphere, or circle. The fact that the numerator and the denominator of the eigenvalue of $S(k)$ above are complex conjugate of each other exhibits the unitarity of $S(k)$ for $k > 0$. The poles of $S(k)$ are the roots of $H^{(1)}_{p+\ell}(ka)$ which occur when k is complex. In particular, for $n = 2$, consider an arbitrary continuous function on the circle of radius a. Expanding it in Fourier series, one obtains

$$S(k)\left(\sum_{-\infty}^{\infty} \alpha_n e^{in\phi_0}\right) = -\sum_{-\infty}^{\infty} \alpha_n \frac{H_n^{(2)}(ka)}{H_n^{(1)}(ka)} e^{in\phi_0} , \tag{3.21}$$

where (a,ϕ_0) denotes a point on the circle. If k is complex, $S(k)$ has complex poles in the lower half-plane of the complex wavenumber plane.

Since the complex singularities determined by integral equations depend only upon the homogeneous integral equation, it suffices to consider the simple boundary value problem

$$(\Delta + k^2)u^s(\bar{r}) = 0 , \ \bar{r} \in \Omega . \tag{3.22}$$

$$u^s(\bar{r}_0) = -e^{-ikacos\phi_0} , \ \bar{r}_0 \in \Gamma .$$

and

$$\lim_{r\to\infty} \sqrt{r} \left(\frac{\partial}{\partial r} - ik\right)u^s(\bar{r}) = 0 .$$

This has the known separation of variable solution given by

$$u^s(\bar{r}) = -\sum_{-\infty}^{\infty} (-i)^n \frac{J_n(ka)}{H_n^{(1)}(ka)} H_n^{(1)}(kr)e^{in\phi} . \tag{3.23}$$

This solution exhibits the same complex singularities as the scattering matrix. The Brakhage-Werner representation [7]

$$u^s(\bar{r}) = \frac{ika}{4}\int_0^{2\pi}\left(\frac{\partial}{\partial(kn')} - i\tau\right) H_0^{(1)}(kR)\mu(a,\phi')d\phi' . \qquad (3.24)$$

leads to the integral equation:

$$\mu(a,\phi_0) + \frac{ika}{4}\int_0^{2\pi}\left(\frac{\partial}{\partial(ka)} - i2\tau\right) H_0^{(1)}(kR_0)\mu\, d\phi' \qquad (3.25)$$

$$= -2\, e^{-ika\cos\phi_0} .$$

The unique solution of this integral equation is

$$\mu(a,\phi_0) = \frac{2}{\pi ka}\sum_{-\infty}^{\infty}\frac{(-i)^{n-1}}{H_n^{(1)}(ka)}\;\frac{J_n(ka)\, e^{in\phi_0}}{[J_n'(ka) - i\tau J_n(ka)]} . \qquad (3.26)$$

Again the complex singularities are the same as those of the scattering matrix.

The SEM type of representation for this problem would be

$$u^s(\bar{r}) = \frac{ika}{4}\int_0^{2\pi}\frac{\partial}{\partial(kn')} H_0^{(1)}(kR)\, \nu(a,\phi')d\phi'$$

and the corresponding integral equation would be

$$\nu(a,\phi_0) + \frac{ika}{4}\int_0^{2\pi}\frac{\partial}{\partial(ka)} H_0^{(1)}(kR_0)\nu\, d\phi'$$

$$= -2\, e^{-ika\cos\phi_0}$$

with a solution given by

$$\nu(a,\phi_0) = \frac{2}{\pi ka} \sum_{-\infty}^{\infty} \frac{(-i)^{n-1} J_n(ka)}{J_n'(ka) H_n^{(1)}(ka)} e^{in\phi_0} \tag{3.29}$$

Here, in addition to the complex singularities given by the scattering matrix, the zeros of $J_n'(ka)$ occur at those values of k which correspond to the eigenvalues of the corresponding interior homogeneous Neumann problem. They are an exampl of the interior resonances of SEM and they do not appear in the separated solution for the scattered wave. They are clearly seen to be method dependent.

4. Additional Comments on the SEM

Eigenmode expansions and other spectral properties have been employed in the development of the SEM formalism by C. Baum [2,3,4] and L. Marin and R. Latham [18]. While these have been used only formally, even the formal expressions are often meaningless. For example, the left and right eigenvalues are defined formally by

$$<\tilde{\underline{\Gamma}}(\overline{r},\overline{r}';s);\overline{R}_\beta(\overline{r}';s)> = \overline{\lambda}(s)\overline{R}_\beta(\overline{r};s) ,$$

$$<\overline{L}_\beta(\overline{r};s);\tilde{\underline{\Gamma}}(\overline{r},\overline{r},';s)> = \overline{\lambda}(s)\overline{L}_\beta(\overline{r};s)$$

where $\tilde{\underline{\Gamma}}(r,r';s)$ represents some integral operator involving a free space dydadic kernel. Then, the integral equation

$$\langle \tilde{\underline{\Gamma}}(\bar{r},\bar{r}';s);\ \bar{J}(\bar{r}';s)\rangle = \tilde{\underline{I}}(\bar{r};s)$$

has a formal solution given by

$$\bar{J}(\bar{r};s) = \sum_{\rho} \frac{1}{\bar{\lambda}_{\rho}(\delta)} \frac{\langle \bar{\tilde{L}}_{\rho}(\bar{r};s);\ \tilde{\underline{I}}(\bar{r}';s)\rangle}{\langle \bar{L}_{\rho}(\bar{r};s);\ \bar{R}_{\rho}(r';s)\rangle} \bar{R}_{\rho}(\bar{r};s) .$$

Since neither properties of the integral equation, nor the space in which its solutions are sought are specified, it is difficult to comment on the validity of this procedure. If the space were a Hilbert space of square integrable functions and the integral equation were of the first kind with a hermitian symmetric Hilbert-Schmidt kernel then under a strong convergence condition it is valid. (Cf.e.g., the discussion in Tricomi [29]).

An alternate and more promising interpretation would be in terms of non-self adjoint operators as they are, for example, discussed in Ramm [21.1] . This interpretation, however, requires a knowledge of dissipative operators and root spaces and consequently will not be further commented upon here.

It appears that much of the doubtful SEM formalism arises whenever a formal limit of a matrix relation is taken. While all matrices have eigenvalues and associated invariant spaces, this is not true of their limit, integral operators, and, even if the integral operator does have a eigenvalues there may not be enough to span the range of the operator. As a case in point, the homogeneous MFIE has only one eigenvalue, namely "-1" but this value can arise for many different values of k-those corresponding to both interior and exterior resonances. While these k's corresponding to exterior resonanaces are eigenvalues of the exterior scattering problem for the vector wave equation, they are not eigenvalues of the MFIE.

Since there is a rapidly developing theory of equivalent circuits based on the SEM formalism, this formalism should be carefully evaluated and only that part of it for which a mathematical basis can be furnished should be used. We hope to return to this subject in the future.

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REVIEW

Unedited draft - The general method of eigenvibrations in the theory of diffraction. (by N. Voitovič, B. Kacenelenbaum, A. Sivov
Spectral prop. of diffractions problems (by M.S. Agranovič)
NAUK Moscow, 1977.

The problems of diffraction and scattering of electromagnetic or ccoustic waves have been treated at different levels of physical evidence or intuition and of mathematical rigor. In some engineering publications "the analysis" consisted of little more than formal manipulation without much worry about the spaces in which the operators act, about completeness or convergence. At the other end of the scale sophisticated mathematical theories have been developed which so far failed to permeate to the level of physical applications. The monograph under review is a splendid example of the intermediate "interdisciplinary approach, attempting to open up the communication between the mathematicians and practical engineers working on problems of communication theory, In the opinion of the reviewer their approach appears to be the most promising way of overcoming the vocabulary and traditional formalistic differences in approach to the subject. The first part outlines some eigenvalue-eigenfunction problems of electromagnetic theory, particularly the ε-technique which has been pioneered by Voitovič, Kacenelenbaum, and Sivov. The "generalized method" consists of representing the solutions in terms of eigenfunctions of a homogeneous problem in which the eigenvalue is not the frequency (or a root of the frequency) but some other parameter of the system, such as for example, the dielectric constant of a reference system occupying the same volume. It is not necessary for each term of the series to satisfy the boundary conditions (as is the usual practice in applying the Rayleigh-Ritz or Galerkin procedures). Only the whole series satisfies the imposed boundary conditions. Part two considers the significance of the boundary conditions. The conditions of impedence type (the ω techniques), the adjoint boundary conditions (the ρ-techniques), conditions given at infinity, the metallic surface conditions are some of the topics discussed. Chapter 3 covers the variational techniques. Using the ε-approach they considered the system

$$\Delta u + k^2 \varepsilon u = 0 \text{ в } V^+,$$
$$\Delta u + k^2 u = 0 \text{ в } V^-$$

$$u^+ - u^- |_{S_\varepsilon} = 0,$$
$$\frac{\partial u^+}{\partial N} - \frac{\partial u^-}{\partial N}\Big|_{S_\varepsilon} = 0,$$

$$u|_S = 0.$$

Considering ε as a (fixed) constant, the authors look for the stationary behavior of the following functional

$$K(u) = \frac{\int\limits_V (\nabla u)^2 \, dV}{\int\limits_{V^-} u^2 \, dV + \varepsilon \int\limits_{V^+} u^2 \, dV} \quad (V = V^+ + V^-),$$

$$u = u_n + \mu\varphi,$$
$$K(u) = K(u_n) + O(\mu^2).$$

or of

$$L(u) \int_V (\nabla u)^2 \, dV - k^2 \int_{V^-} u^2 \, dV - k^2 s \int_{V^+} u^2 \, dV.$$

Alternately setting $L(u) = 0$. They consider the funccional $E(u)$ of the ε-technique

$$E(u) = \frac{\int_V (\nabla u)^2 \, dV - k^2 \int_{V^-} u^2 \, dV}{k^2 \int_{V^+} u^2 \, dV}.$$

Variants of these techniques are applied to a variety of problems in Chapter 4. Metallic tuning devices, wave guides constitute typical examples of applications. A separate chapter, Spectral Properties of Diffraction Problems (123 pages plus bibliography), written by Agranovič is modestly called an appendix. It fully deserves a separate review. It contains the introductory definitions concerning non-selfadjoint operators in Hilbert spaces, basis and biorthogonal systems, Sobolev spaces, and elliptic operators. He discusses some properties of elliptic pseudodifferential operators and of elliptic boundary value problems. This preliminary discussion leads to theorems concerning the ellipticity of the boundary value problems in diffraction theory, and the order of the corresponding operators and of their selfadjointness. Problems in diffraction theory are reduced to the study of spectral properties of certain pseudodifferential operators theorems on disipative property of such operators are proved. The author's s-technique is rigorously restated. Invertibility (which was required in certain steps of Chapter 2 is proved as a theorem. Some properties of the s-technique are closely related to the general techniques initiated by M.G. Krein. The appendix does not discuss the details of the variational approach of the authors. Further research is needed to resolve many outstanding issues raised by the authors approach to diffraction theory.

A CRITIQUE OF THE SINGULARITY EXPANSION AND EIGENMODE EXPANSION METHODS

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SUMMARY

The authors outline some available theoretical techniques interpreting the complex singularities of the S matrices, and point out various difficulties which so far have prevented a formulation of a consistent and vigorous theory. Some directions for future research are suggested. These are primarily concerned with the singularity expansion and the eigenmode expansion methods. In particular, some recent results of the Russian school pertaining to the theory of non-self-adjoint operators are summarized.

A CRITIQUE OF THE SINGULARITY EXPANSION AND EIGENMODE EXPANSION METHODS

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INTRODUCTION

The role of complex singularities in scattering problems is a rich and diverse one [see Dolph and Scott (1)]. Baum (2) has been of the prime movers in the United States of the subjects of the title. He has been helped in his efforts by the work of Marin (3) and Tesche (4). In the Soviet Union their counterparts are Voitovič, Kacenelenbaum and Sivov (5). There appears, however, to be an important difference between the two countries in that in the USSR mathematicians, beginning with A. G. Ramm, carefully examined the formalism of the approaches. Such efforts have culminated in a book with an interdisciplinary spirit entitled: The Generalized Methods of Eigen Vibrations in the Theory of Diffraction (5). The book, which also contains an extensive appendix by Agranovič entitled: Spectral Properties of Diffraction Problems, has been reviewed by the first author (with considerable help from the second author). This review should appear shortly in Mathematical Reviews and an effort is underway to have the text itself translated into English. It is fortunate that much of the pertinent Soviet work has appeared in English translation in the Journal of Radio Engineering and Electron Physics. The readers attention should be called to the two papers of Ramm [(6), (7)] and the series of papers by Agranovič [(8), (9), (10)]. All of this work was stimulated by the pioneering paper of Kacenelenbaum (11) and its sequel by Voitovič, Kacenelenbaum and Sivov (12). There also is an important paper by Agranovič and Golubeva (13). [A word of caution must be injected here. In a relevant paper of Golubeva (14), the word "proposition" is translated as "conjecture," which one must admit does change the flavor. Since each "conjecture" is followed by a full proof, the reader must be alert to the translation problems.]

SINGULARITY EXPANSION METHOD(SEM)

The Singularity Expansion Method and its relationship to the mathematical theory of scattering has recently been analyzed by Dolph and Cho (15) and consequently no exhaustive detail on the technique is required here. Instead, a summary of the results of that work will be given.

The Singularity Expansion Method is a generalization of well-known techniques of linear circuit theory in which the singularities of a transfer function are used to determine the transient response by the Heaviside expansion theorem. In electromagnetic theory, the singularities are found by first applying a two-sided Laplace transform, parameter s, to the Maxwell equations and then constructing an integral equation for the scattered field. Complex singularities $\{s_n\}$ appear as poles of the inverse of this equation and are determined from the non-trivial solutions of the corresponding homogeneous integral equation.

The scattering operator is a unitary operator on a Hilbert space. Its Laplace transform, the scattering matrix, is analytic in the right-hand s-plane including the axis and is meromorphic in the left-half plane. It has poles in the left-half plane at those values of s for which there exist non-trivial outgoing solutions of the reduced Maxwell's equations satisfying the boundary conditions. These discrete values of s are complex eigenvalues of the exterior scattering problem. Shenk and Thoe (16) have established a one-to-one correspondence between the poles of the S-matrix and the poles of the integral equation.

Some of these poles occur on sheets of Riemann surfaces and serious difficulties arise in interpreting their meaning. This fact can be deduced from the study of the integral equation. Only the Green's Kernel can be continued analytically into such "forbidden domains". The resolvent can not.

While it is difficult to relate these concepts in general since most of the work in SEM involves a formalism in which neither spaces nor properties of the integral equation are given, this can be done for the papers of Marin (3). He uses a Hilbert space consisting of tangential currents on the surface of a convex body to discuss the magnetic field integral equation. He deduces a Fredholm integral equation of the second kind from this and uses Carleman's Fredholm theory for the determination of the natural modes. The solution of the corresponding homogeneous integral equation contains both exterior and interior resonances, the latter being purely imaginary.

To relate the scattering matrix to the integral equation one makes use of its representation as a compact Fredholm integral operator. The kernel of this operator is a transmission coefficient arising from the asymptotic form of the scattered field. Since its determination involves the solution of the scattering problem, rather than use the above representation, it is simpler to use the one-to-one correspondence between the kernel of this representation and the solution of the vector wave equation. It can be given in terms of a vector integral equation which is more general than the magnetic field integral equation. Fredholm theory is now used. Recall that if A is a compact operator (such as the above mentioned operator defining the integral equation) then the first part of the Fredholm alternative states that $A\phi = f$ has a unique solution ϕ if $A\phi = 0$ has only the trivial solution. Here the first part of the Fredholm alternative yields a unique solution to the general equation for the right-hand s-plane including the imaginary axis. The analytic Fredholm theorem for compact operators which is given, for example, in Reed and Simon (17), then implies the analytic and meromorphic properties of the scattering matrix discussed above.

For the magnetic field integral equation, one must use the second part of the Fredholm alternative, namely: If $A\phi = 0$ has non-trivial solutions then:

(i) $A\phi = 0$, $A^*\psi = 0$, where A^* is the adjoint of A, have the same finite number of solutions.

(ii) For $A\phi = f$ to have a solution ϕ_f, f must be orthogonal to all of the solutions of $A^*\psi = 0$.

Since the solution to $A\phi = 0$ can be added to ϕ_f, then clearly the process is not unique and no one-to-one map between the scattering matrix and the integral equation exists for the same half-plane. The non-trivial solutions of the corresponding homogeneous integral equation occurring for complex $\{s_n\}$ do, however, correspond to poles of the scattering matrix but the interior resonances corresponding to purely imaginary $\{s_n\}$ are not poles of the scattering matrix nor do they appear in the solution for the scattered field. They are consequently method dependent. The exterior resonance corresponding to complex $\{s_n\}$ are method independent and intrinsic to the scattering body.

Since a one-to-one correspondence fails to exist, it is conceivable that there are complex poles of the scattering matrix which are not given by the solutions of the homogeneous magnetic integral equation corresponding to complex $\{s_n\}$. No examples, however, in which this occurs are known and thus the problem remains open.

Since it is generally believed that the scattering matrix contains all observable information about the scattering process, the above observations substantiate the claim that the complex singularities are intrinsic to the scatterer whereas those which are purely imaginary are not.

Further difficulty arises in that most problems treated by the SEM formalism invole numerical techniques applied to matrix equations obtained from finite approximations to the integral equations. Hence, Hilbert space solutions are not really appropriate. Fortunately, the results can be given in terms of solutions in the Banach space of continuous functions. In this space the Fredholm theorem as given by Steinberg (18) can be used to discuss the magnetic field integral equation and to establish the one-to-one map between the scattering matrix and the associated set of vector integral equations. This map between the scattering matrix and the integral equation has been carried out for convex, perfectly conducting bodies only [see Lax and Phillips (19), Brakkage and Werner (20), and Dolph and Cho (15)]. Certain other aspects of the SEM formalism have also been investigated by Dolph and Cho (15) and serious doubts were raised regarding those parts of it where integral equations of the first kind are used. Also, there appears to be a confusion in the SEM literature between the eigenvalues of the integral equations and those of the vector wave equations and a belief persists that what is true for the finite matrix equations holds in the limit.

Though said elsewhere, it seems worth repeating that areas in which the formalism needs further investigation include:

I. The construction of variational principles useful for providing estimates for the location of the poles in the SEM and possibly of use in establishing their existence for off-axis poles not covered by the known Lax-Phillips results (21) for the scalar case and their generalization by Beale (22) for the electromagnetic case.

II. An investigation of integral equations of the first kind as used in SEM and a possible justification of their use through regularization methods, such as those of Tihonov (23).

III. The extension to the electromagnetic case of theorems sufficient to guarantee that all poles are simple. This should include generalizations of the theorems of Steinberg (18) and Howland (24).

IV. An investigation of the entire functions which arise, through the Mittag-Leffler theorem, in the formalism. This should include the determination of conditions under which they do not occur and their explicit form when they do occur.

V. The creation of a systematic theory of the asymptotic contribution of branch lines. As can be seen from the discussion in Dolph-Scott (1), such a theory could have implications in many areas.

At this point a few remarks on the T-matrix would seem to be in order. The T-matrix formalism for scattering has proved to be an efficient way of obtaining numerical results in a number of complicated problems [see Bolomey and Wirgin (25)]. It was used by Waterman (26) in 1969 and subsequent publications include papers by Peterson and Ström[(27), (28), (29)], Ström [(30), (31), (32)] and Varatharajulu and Pao (33). However there is a relationship between the T and S matrices, namely $T = S - 1$, as stated in Wu and Ohmura (34), and physicists in the past have used S in preference to T.

EIGENMODE EXPANSION METHOD

Baum (2) has advanced the idea of synthesizing transient responses by means of the eigenmodes of integral equations of the first kind describing the system response. Considerable caution must be exercised in this approach however, since as Ramm [(6) (7)] has pointed out, incorrect results can sometimes arise due to the non-self-adjoint property of the operators that are treated. At this point it appears certain that the theory of non-self-adjoint operators can be used to contribute substantially to the understanding and limitation of the formalism of EEM (and SEM). While the idea of using non-self-adjoint operator theory in scattering problems is not new---it was already suggested by Dolph (35) in 1960--it does not appear to have been used in connection with SEM and EEM in the English literature. This is rather surprising since this theory has been imployed in connection with scalar diffraction problems in papers translated from the Russian and reprinted in the journal Radio Engineering and Electron Physics beginning after the formalism presented in the paper by Voĭtovič, Kacenelenbaum and Sovov (12).

While this latter paper contained a formalism similar to EEM for both the scalar and electromagnetic case, including dielectric problems, its subsequent interpretat at least in translated papers known to the writers, in terms of non-self-adjoint operator theory have been limited to the scalar problem. In the simplest case one attempts to construct a formal solution to the system

$$(\Delta + k^2)u = f \tag{1}$$

$u = 0$, on a smooth convex scatterer Γ,

satisfying the radiation condition as a series $u = u_0 + \Sigma A_n \phi_n(\underline{x})$. Here u_0 is the incident field and $\{\phi_n(\underline{x})\}$ are the eigenfunctions of a compact integral operator A, which for in three dimensions, is given explicitly by

$$A\phi_n = \int_\Gamma G_0(\underline{x}-\underline{y})\phi_n(\underline{y})d\sigma = \lambda_n(k)\phi_n(\underline{x}) , \tag{2}$$

where the free space Green's function G_0 is

$$G_0 = \frac{1}{4\pi}\frac{e^{ik|\underline{x}-\underline{y}|}}{|\underline{x}-\underline{y}|} . \tag{3}$$

It was further assumed that the coefficients in the above expansion could be determ by the Fourier coefficient formula

$$A_n = \frac{\int_\Gamma (u-u_0)\phi_n d\sigma}{\int_\Gamma \phi_n^2 d\sigma} \tag{4}$$

Ramm (7) interpreted and clarified these results by using a Hilbert space $L_2(\Gamma)$ with the usual Hermitian inner product

$$(f,g) = \int_\Gamma f(\underline{x})\bar{g}(\underline{x})d\sigma \tag{5}$$

Since G_0 is real symmetric but complex valued, this implies that

$$(A\phi,\psi) = (\phi,\bar{A}\psi) \tag{6}$$

so that the operator A is non-self-adjoint.

Since the compact operator A is non-self-adjoint, it may have root vectors instead of simple eigenvectors. That is for a given λ, there may exist an integer $p > 1$ such that $(A-\lambda I)^p\phi = 0$ for some ϕ, while $(A-\lambda I)^q\phi \neq 0$ for all $q < p$. (In the matrix case this happens when non-simple elementary divisors occur and requires the use of the *Jordan normal form rather* than the diagonal form in the canonical representation of the matrix.) However, Ramm was able to show that while the system of root vector was always complete in $L_2(\Gamma)$, the simple form of the coefficients given above would only occur if the surface Γ were such that A defined over it was a normal operator, that is $AA^* = A^*A$ (this condition is necessary and sufficient for A to be a diagonal operator). The normality of the operator can be tested as follows: For example, consider the operator $A: H \to H$, where

$$Af = \frac{1}{4\pi}\int_\Gamma \frac{\exp(ik|s-t|)}{|s-t|} f(t)dt \tag{7}$$

A is normal provided the integral

$$\int_\Gamma \frac{\sin(k|x-t| \; |y-t|)}{|x-t| \; |y-t|} d\sigma \; , \; x,y\epsilon\Gamma \tag{8}$$

vanishes, a condition that Ramm was able to show was true if Γ is a sphere. In complicated problems it may have to be tested by direct computation.

If the operator A is not normal, considerable complexity arises in the theory and computational schemes. The problem is that in general one has to work in an infinite dimensional analog of the Jordan normal form, since A fails to be diagonalizable. The lemma of Schur [see Gohberg and Krein (36)] states that any completely continous operator A mapping a Hilbert H space onto itself can be represented in a triangular form. Specifically, there exists an orthogonal basis ω_j of H such that

$$A\omega_j = \sum_{i\leq j} a_{ji}\omega_i \tag{9}$$

where $a_{ji} = (A\omega_j,\omega_j) = \lambda_j$, (3) denoting inner product and λ_j being an eigenvalue of A.

There are many areas which need detailed mathematical investigation. For example Voĭtovič, Kacenelenbaum and Sivov consider several variational principles which produce stationary solutions. These are reminisicent of those of Schwinger and McFarlane, the latter occuring in the problem of anomalous propagation through the atmosphere. Attempts to make mathematical sense of these occur in Dolph (37), Dolph and Ritt (38) and Dolph, McLaughlin and Marx (39). The max-min characterizat of the last paper unfortunately depends upon the dimension of the approximating space and so it badly needs reformulation. The only other pertinent work it seems to be that of Morawetz (40) where as discussed in Dolph-Scott (1) a variational principle is given implicitly.

In connection with variational principles, the following facts should be mentioned. The variational approach given in (5) has been briefly reviewed by Dolph in (41). Voĭtovič, Kacenelenbaum and Sivov introduced the ε-approach by considering an approximate scattering problem (the ε-problem) and the corresponding systems of equations

$$
\begin{aligned}
\Delta u + k^2 \varepsilon u &= 0 \quad \text{in } V^+ \\
\Delta u + K^2 u &= 0 \quad \text{in } V^- \\
u^+ - u^- \big|_{S_\varepsilon} &= 0 \\
u \big|_S &= 0 \\
\frac{\partial u^+}{\partial n} - \frac{\partial u^-}{\partial n} &= 0 \big|_{S_\varepsilon} ,
\end{aligned} \tag{10}
$$

leading to a miminization of the functional

$$
L(u) = \int_V (\nabla u)^2 dV - k^2 \int_{V^-} u^2 dV - k^2 \varepsilon \int_{V^+} u^2 dV \tag{11}
$$

A rigorous justification of such techniques still remains an open problem. Similar variational principles were given by J. Schwinger and H. Levine (42). [Also see Kato (43), and Dolph (37)]. As in the case discussed by Dolph and Ritt (38), or Dolph (37), it can be conjectured that the real and imaginary parts of the unknown function near the stationary point lie on a saddle-like surface, but the orientation of such a saddle is unknown.

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ZEROS OF THE INPUT IMPEDANCE FUNCTION FOR THIN CYLINDRICAL AND PROLATE SPHEROIDAL ANTENNAS

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1. Introduction

The input impedance function of the thin cylindrical antenna has been studied by Hallen [1], Schelkunoff [2], King-Middleton [3], Tai [4] and others. Although the methods used by these authors are different from each other, yet they yield the numerical results for the input impedance in the (real-valued) frequency domain which agree relatively closely not only with each other but also with experimental results.

The main objective of our study is to numerically compute the distribution patterns of zeros of the input impedance functions in the complex frequency domain of thin cylindrical and prolate spheroidal antennas, and study their implications. This is carried out by the use of Schelkunoff's perturbation method [2] with a slight modification. The Schelkunoff's method is essentially a heuristic one and so is our present study because we adopt the method formally without establishing a rigorous mathematical justification.

In section 2 the input impedance functions are derived for a cylindrical antenna and a prolate spheroidal antenna and, in section 3, the numerical results are presented for zeros of these functions in the complex frequency plane. In section 4 we offer a concluding remark on our numerical results.

2. Input Impedance

In this section we derive the input impedance functions for thin cylindrical and prolate spheroidal antennas based on the Schelkunoff's method. For convenience of the reader, we will show the essence of the method in some details for the cylindrical case; for the prolate spheroidal case, we will merely show the result since the same procedure is used.

2.1 Cylindrical antenna

Consider a cylinder of length 2ℓ and radius a. By regarding it as a nonuniform transmission line with the source located at the center of the cylinder, we define the line impedance, line admittance, and characteristic impedance at r, $0 \leq r \leq \ell$, as following:

$$Z(r) = jk \frac{\eta_o}{\pi} \ln\left(\frac{2r}{a}\right)$$

$$Y(r) = jk\pi/\eta_o \ln\left(\frac{2r}{a}\right),$$

and

$$K(r) = \frac{\eta_o}{\pi} \ln\left(\frac{2r}{a}\right)$$

where $\eta_o = 120\pi$ ohms and $k = \omega\sqrt{\mu_o \epsilon_o}$.
We then define the average characteristic impedance of the non-uniform line as

$$K_o = \frac{\eta_o}{\pi} \frac{1}{\ell} \int_o^{\ell} \ln\left(\frac{2r}{a} \alpha\right) dr, \qquad (1)$$

where α is some positive real constant number yet to be determined. Carrying out the integral, we get

$$K_o = \frac{\eta_o}{\pi} \ln\left(\frac{2\ell}{a} \frac{1}{\beta}\right), \qquad (2)$$

where $\beta = e/\alpha$. If we set $\beta = e$, (2) reduces to the average characteristic impedance defined by Schelkunoff:

$$K_o\Big]_{\beta=e} = \frac{\eta_o}{\pi} \ln\left(\frac{2\ell}{ae}\right),$$

which can be written in terms of the bicone parameter $\theta_o = a/\ell$ [5], the half-angle of the bicone, as

$$K_o\Big]_{\beta=e} = \frac{\eta_o}{\pi} \ln\left(\frac{2}{e\theta}\right).$$

This shows that the average characteristic impedance of the cylinder corresponds to the characteristic impedance of a bicone whose half-angle is $e\theta_o \simeq 2.7\ \theta_o$. On the physical ground, it seems more appropriate to have the bicone inscribed in cylinder. It is for this reason that we introduced a new averaging procedure in the form of (1). If $\beta = 1$ is chosen, (2) becomes

$$K_o = \frac{\eta_o}{\pi} \ell n\left(\frac{2}{\theta_o}\right) = Z_c(\theta_o) \tag{3}$$

$Z_c(\theta_o)$ being the characteristic impedance of a biconical antenna with the half-angle $\theta_o = a/\ell$.

The average line impedance, Z_o, can also be defined similarly. Thus

$$Z_o = jk \frac{\eta_o}{\pi} \frac{1}{\ell} \int_o^{\ell} \ell n\left(\frac{2r}{a} \alpha\right) dr$$

$$= jkK_o. \tag{4}$$

Following Schelkunoff, we set

$$Y_o = Z_o/K_o^{\ 2} \tag{5}$$

and

$$k = \sqrt{Z_o Y_o}.$$

Note that Y_o is not the average admittance in the sense of (4). We now denote the measure of nonuniformity of the line by

$$\hat{Z}(r) = Z(r) - Z_o, \tag{6}$$

$$0 < r < \ell$$

$$\hat{Y}(r) = Y(r) - Y_o. \tag{7}$$

Then, we can write

$$\hat{Z}(r) = jk \frac{\eta_o}{\pi} \ell n\left(\frac{r}{\ell}\right), \quad 0 \leq r \leq \ell. \tag{8}$$

This "perturbation" of line impedance is awkward because it tends to $- j\infty$ as $r \to 0$. This is a consequence of the definition of the line impedance as given previously, and is one of the mathematical difficulties in the Schelkunoff's method. We then assume that the nonuniform line varies in such a way that the following restriction holds throughout the line:

$$\frac{\hat{Y}(r)}{\hat{Z}(r)} = - \frac{Y_o}{Z_o} \tag{9}$$

This condition is necessary for linearization of the theory.

By the nonuniform transmission line theory, the line voltage and current at r, $0 \le r \le \ell$, are given by

$$V(r) - V_o(r) - \frac{1}{K_o}\int_o^r \hat{z}(t)[jV(t)\sin k(r-t)$$
$$+K_o I(t)\cos k(r-t)]dt \quad (10)$$

and

$$I(r) = I_o(r) + \frac{1}{K_o^2}\int_o^r \hat{z}(t)[jK_o I(t)\sin k(r-t)$$
$$+V(t)\cos k(r-t)]dt, \quad (11)$$

where

$$V_o(r) = V_o\cos kr - jK_o I_o \sin kr, \quad (12)$$

$$I_o(r) = I_o\cos kr - j\frac{V_o}{K_o}\sin kr, \quad (13)$$

$$I_o = I(o),\ V_o = V(o).$$

$V_o(r)$ and $I_o(r)$ denote, respectively, the line voltage and current of a uniform transmission line. Eqs. (10) and (11) are coupled Volterra integral equations of the second kind for I(r) and V(r). Since such Volterra integral equations in L_2 possess unique solutions and the operators are contractive, one can apply Piccard's iteration to obtain a convergent solution. Thus, we formally write the solutions of (10) and (11) as

$$V(r) = \sum_{n=0}^{\infty} V_n(r) \quad (14)$$

$$I(r) = \sum_{n=0}^{\infty} I_n(r) \quad (15)$$

For n = 0, $V_o(r)$ and $I_o(r)$ are given by (12) and (13); for $n \ge 1$,

$$V_n(r) = -\frac{1}{K_o}\int_o^r \hat{z}(t)[j\,V_{n-1}(t)\sin k(r-t)$$
$$+ K_o I_{n-1}(t)\cos k(r-t)]dt, \quad (16)$$

$$I_n(r) = \frac{1}{K_o^2}\int_o^r \hat{z}(t)\,[jK_o I_{n-1}(t)\sin k(r-t)$$
$$+ V_{n-1}(t)\cos k(r-t)]dt \quad (17)$$

Unfortunately, it becomes impractical to compute the iterates beyond the first and this is a major difficulty of this theory. We are then forced to accept the "approximate" solutions as

$$V(r) \cong V_o(r) + V_1(r), \tag{18}$$

$$I(r) \cong I_o(r) + I_1(r). \tag{19}$$

Following Schelkunoff, we set the terminal impedance at $r = \ell$ as

$$Z(k\ell) = \frac{V(k\ell)}{I(k\ell)} \simeq \frac{V_o(k\ell) + V_1(k\ell)}{I_o(k\ell) + I_1(k\ell)} . \tag{20}$$

We note that $V_1(k\ell)$ and $I_1(k\ell)$ in (20) are the first iterated solutions of the Fredholm integral equations with r replaced by ℓ in (10) and (11). As such, unlike the Volterra integral equations, we no longer have a priori assurance of the uniqueness of the solutions, nor of the convergence of the iterated solutions. Even if we assume that the iterated solutions of the forms (18) and (19) are acceptable approximate solutions for $k > 0$, (i.e., for the real-valued frequencies), it is possible that such an assumption may not be valid when analytically continued in the complex frequency plane as we must eventually do for the computation of zeros of the input impedance function. In spite of these unsettled mathematical questions, we formally proceed and postulate

$$Z(k\ell) = \frac{K_o^{\,2}}{Z_a(k\ell)} , \tag{21}$$

where

$$Z_a(k) = K_o^{\,2} Y(k\ell). \tag{22}$$

$Y(k\ell)$ [5] represents the terminal admittance of a biconical antenna with which our cylindrical antenna is associated. The configuration of the biconical antenna depends on the choice of the value of β. Evidently, the postulations (21) and (22) are heuristic. Equating (20) and (21), we can find the input impedance function of the cylindrical antenna at $r = 0$. This is the Schelkunoff's theory of the cylindrical antenna. Actually, Schelkunoff approximated (20) as

$$Z(k\ell) \approx \frac{V_o(k\ell) + V_1(k\ell)}{I_o(k\ell)}$$

At this point, we depart from the Schelkunoff's theory and define a complex variable s by

$$s = jk\ell \tag{23}$$

and formally replace $k\ell$ in (20) and (21) by $-js$. After some lengthy manipulation, we obtain the desired complex-valued input impedance of the cylindrical antenna in s domain and write

$$Z_{in}^{(c)}(s) = K_o \frac{F_c(s)}{G_c(s)}, \tag{24}$$

where

$$F_c(s) = f(s) - \frac{1}{2\Omega - 4\ell n\beta}[e^s E(2s) + e^{-s}E(-2s)] - \frac{y(s)}{2\Omega}[e^s E(2s) - e^{-s}E(-2s)], \tag{25}$$

$$G_c(s) = g(s) - \frac{1}{2\Omega - 4\ell n\beta}\left\{4\ell n\beta \sinh(s) - [e^s E(2s) - e^{-s}E(-2s)]\right\} - \frac{y(s)}{2\Omega}\left\{4\ell n\beta \cosh(s) - [e^s E(2s) + e^{-s}E(-2s)]\right\}, \tag{26}$$

$$f(s) = \frac{e^s}{2}[1 + y(s)][1 + \Gamma(s)e^{-2s}], \tag{27}$$

$$g(s) = \frac{e^s}{2}[1 + y(s)][1 - \Gamma(s)e^{-2s}], \tag{30}$$

$$y(s) = \frac{1}{2\Omega}\left\{\begin{array}{l} \ell n(2)\cdot[e^{2s} - e^{-2s}] \\ + (2 + e^{2s})E(2s) \\ + e^{-2s}E(-2s) \\ - e^{2s}E(4s) \end{array}\right\}, \tag{31}$$

$$E(\pm s) = \int_o^{\pm s} \frac{1 - e^{2u}}{u} du \tag{32}$$

and

$$\Omega = 2\ln(\frac{2}{\theta_o}) . \tag{33}$$

The input impedance of the biconical antenna of the half-angle θ_o is given by (cf[5])

$$Z_{in}^{(b)}(s) = Z_c(\theta_o) \frac{f(s)}{g(s)}$$

$$= Z_c(\theta_o) \frac{1 + \Gamma(s)e^{-2s}}{1 - \Gamma(s)\ e^{-2s}} \tag{34}$$

The extent of the validity of the input impedance function (24) can be tested by the initial- and final-value theorems. Our analysis shows that, with the applied unit step voltage, the final-value theorem is satisfied but not the initial-value theorem. In fact, we get

$$\lim_{|s| \to \infty} sK_o I(s) \to -1,$$

which is clearly absurd. This finding cast some serious doubt about the numerical values of the zeros of the input impedance function.*

2.2 Prolate spheroidal antenna

We now consider a thin prolate spheroidal antenna inscribed in the cylinder described in section 2.1. By the same procedure, we can obtain the formal expression for the input impedance function in s. The result is

$$Z_{in}^{(s)}(s) = K_o \frac{F_s(s)}{G_s(s)} , \tag{35}$$

where

$$F_s(s) = F_c(s) + \frac{D_1(s)}{\Omega - 2\ln\beta} + \frac{y(s)}{\Omega} jE_1(s) \tag{36}$$

*The approximate impedance function given by eq. (8) by Hallen [1] also fails to meet the initial-value theorem.

$$G_s(s) = G_c(s) - \frac{jE_1(s)}{\Omega - 2\ell n\beta} - \frac{\gamma(s)}{\Omega} D_1(s), \tag{37}$$

$$D_1(s) = - \sinh(s)\sinh(2s)\cdot\ell n(2)$$
$$- \frac{1}{2}\cosh(s)\ [e^{2s}E(2s) + e^{-2s}E(-2s)]$$
$$+ \frac{1}{4}\ [e^{3s}E(4s) + e^{-3s}E(-4s)], \tag{38}$$

$$jE_1(s) = - \cosh(s)\sinh(2s)\cdot\ell n(2)$$
$$- \frac{1}{2}\sinh(s)[e^{2s}E(2s) + e^{-2s}E(-2s)]$$
$$+ \frac{1}{4}\ [e^{3s}E(4s) - e^{-3s}E(-4s)]. \tag{39}$$

3. Numerical results

We have numerically computed zeros of the complex-valued input impedance functions (24) and (35) in the s-plane with $\beta = 1$ for $a/\ell = 0.01$ and 0.1, respectively and the results are shown in Figs. 1 and 2. In Fig. 1, we also show corresponding zeros of the biconical antenna for a direct comparison. As seen from this graph, the zeros of the input impedance function for the cylindrical antenna are only slightly displaced from those for the inscribed biconical antenna. For the purpose of attempting to understand the implications of the numerical results of our present study, we also include in the graph complex-valued poles of the axial current density function on the cylinder of the dimension $a/\ell = 0.01$. The data shown in Fig. 1 are obtained by Tesche [6] by means of the singularity expansion method or the integral equation method; Fig. 1-1 shows the distribution pattern of these complex poles. The SEM complex poles correspond to complex-valued zeros of the input impedance function. We see from Fig. 1 that the distribution pattern of our results and that of the SEM are entirely different. In particular, the zeros of the input impedance based on the Schelkunoff's theory are located only on two "layers", whereas the Tesche's poles occur in multi-layers of similar trajectory. It should be emphasized, however, that the nature of our present problem and that of the SEM are different: our cylinder is a center-fed transmitting antenna, whereas the cylinder in the SEM analysis is a scattering body. Therefore, there is no basis to expect two sets of data to have similar variations.

In Fig. 2, we present zeros of the input impedance function of the prolate spheroidal antenna of the dimension $a/\ell = 0.1$ and the corresponding SEM results obtained by Marin[7]; the distribution pattern of the latter is shown in Fig. 2-1. The zeros of the input impedance function of our prolate spheroidal antenna occur again only on two layers, but the layers now cross each other. Such a crossing of two layers also occur in the case of a biconical antenna when the half angle θ_o is relatively large. Again our numerical results and those of the SEM are widely different.

4. Conclusion

The objective of our present study was to shed some light on the transient behaviors of thin cylindrical and prolate spheroidal antennas by the use of the classical pole-zero method of the transfer function. Unfortunately, however, we do not feel that we can draw definite conclusions about the implication of the numerical results for the zeros of the input impedances of the antennas, mainly because we are not certain, as discussed in section 2.1, whether or not the formal extension of Schelkunoff's theory to the complex frequency domain is meaningful, even though Schelkunoff's theory is generally believed to yield the input impedance function of a thin antenna structure in the sense that the numeral results agree reasonably well with experimental ones in the real frequency domain. Finally, we remark that two similar sets of zeros of the input impedance function in the complex frequency domain such as those for the cylindrical and biconical antennas in Fig. 1 do not necessarily imply that the impedance functions in the real frequency domain are close to each other. For example, the maxima and the minima of the real and imaginary parts of the input impedance of the biconical antenna and those of the cylindrical antenna in Fig. 1 are approximately 2 to 1 in ratio. Conversely, two input impedance functions which agree arbitrarily closel with each other in the real frequency domain may have quite different sets of zeros in the complex frequency domain.

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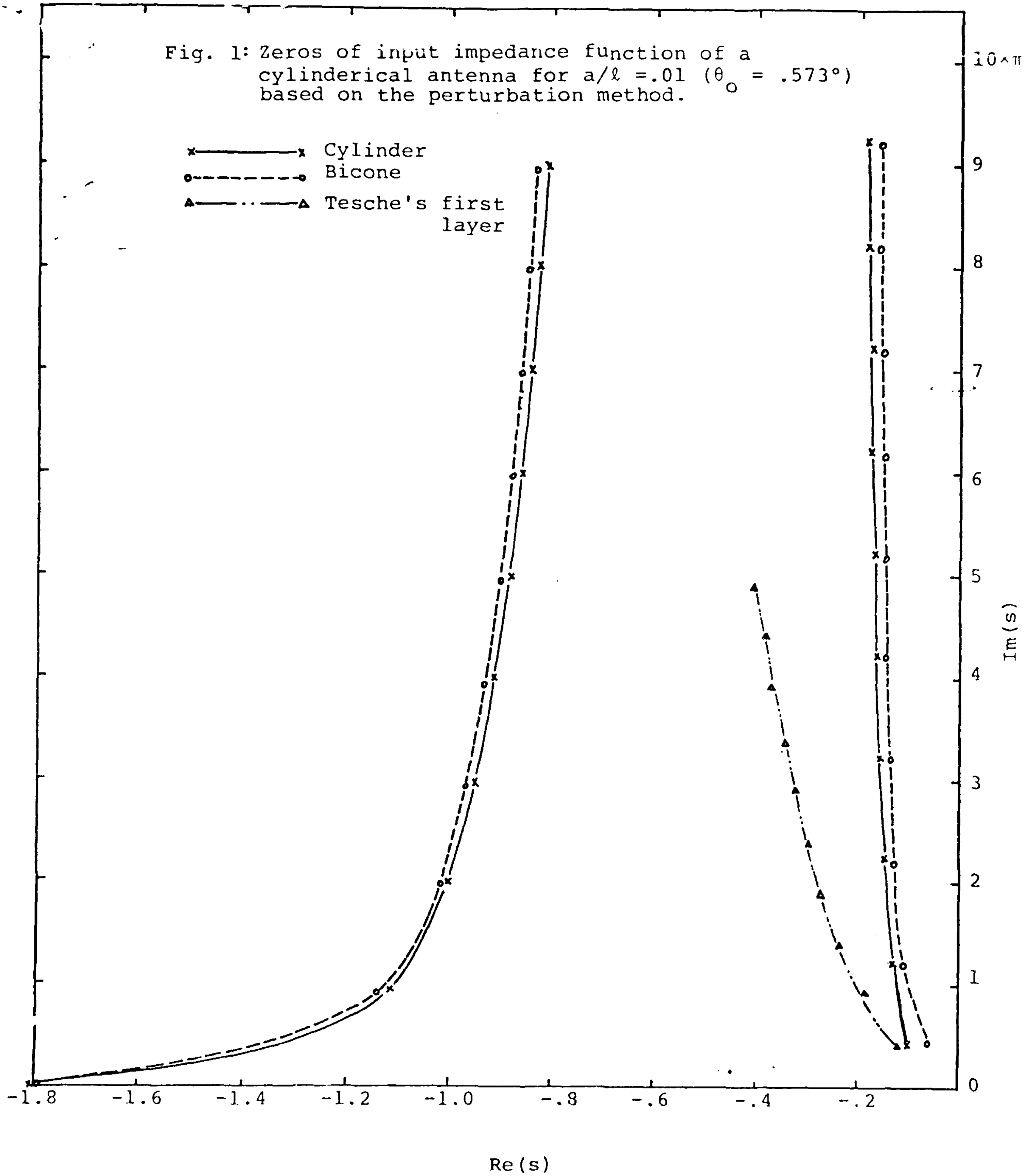
Fig. 1: Zeros of input impedance function of a cylinderical antenna for a/ℓ =.01 (θ$_o$ = .573°) based on the perturbation method.
Cylinder
Bicone
Tesche's first layer
10×π
9
8
7
6
5
4
3
2
1
0
Im(s)
-1.8
-1.6
-1.4
-1.2
-1.0
-.8
-.6
-.4
-.2
Re(s)

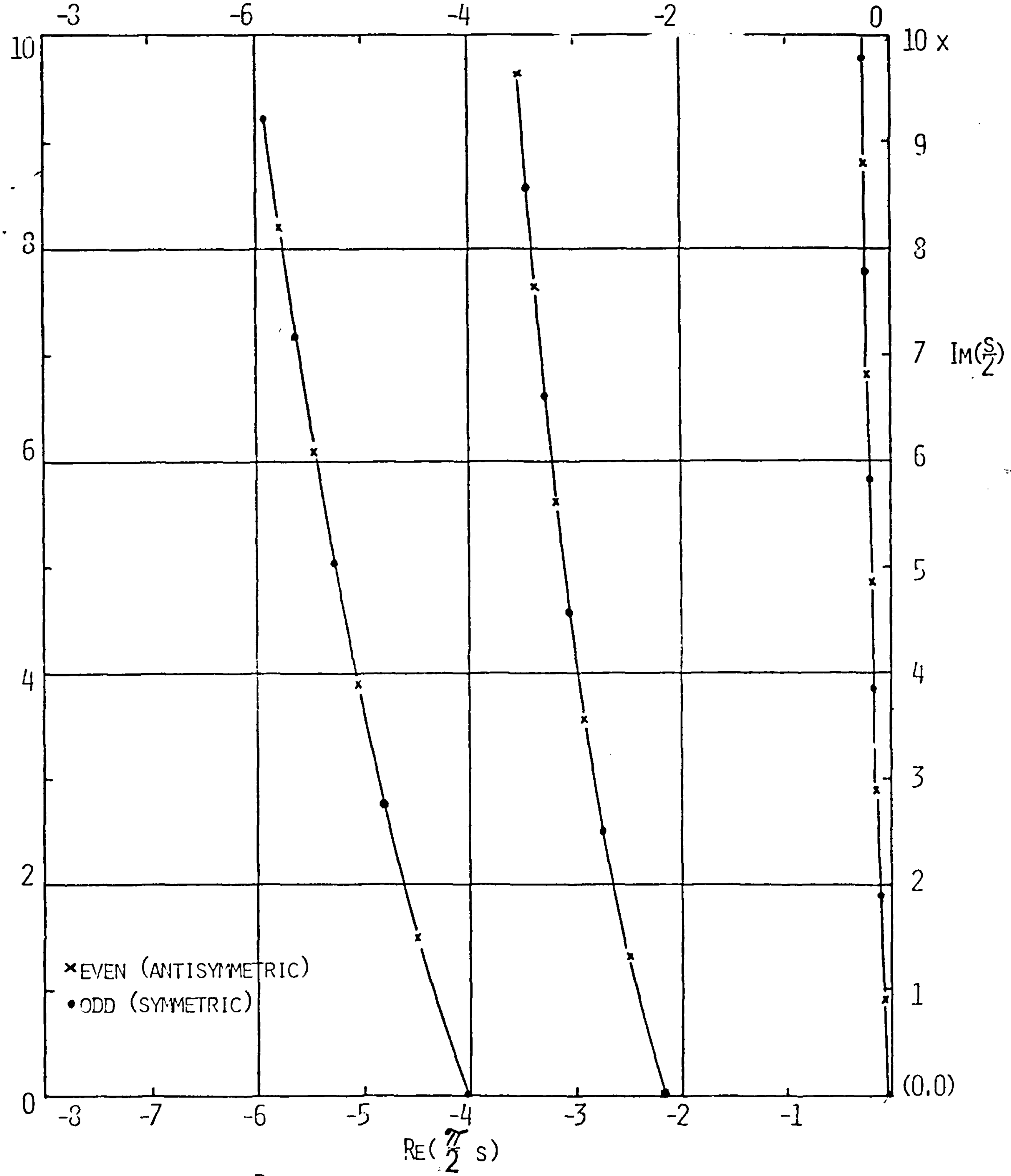

Fig. 1-1: POLES OF THE INDUCED CURRENT FUNCTION OF A THIN CYLINDRICAL SCATTERER OF $a/\ell = 0.01$, BASED ON THE SEM. (AFTER B. SINGARAJU B. GIRI, C. BAUM)

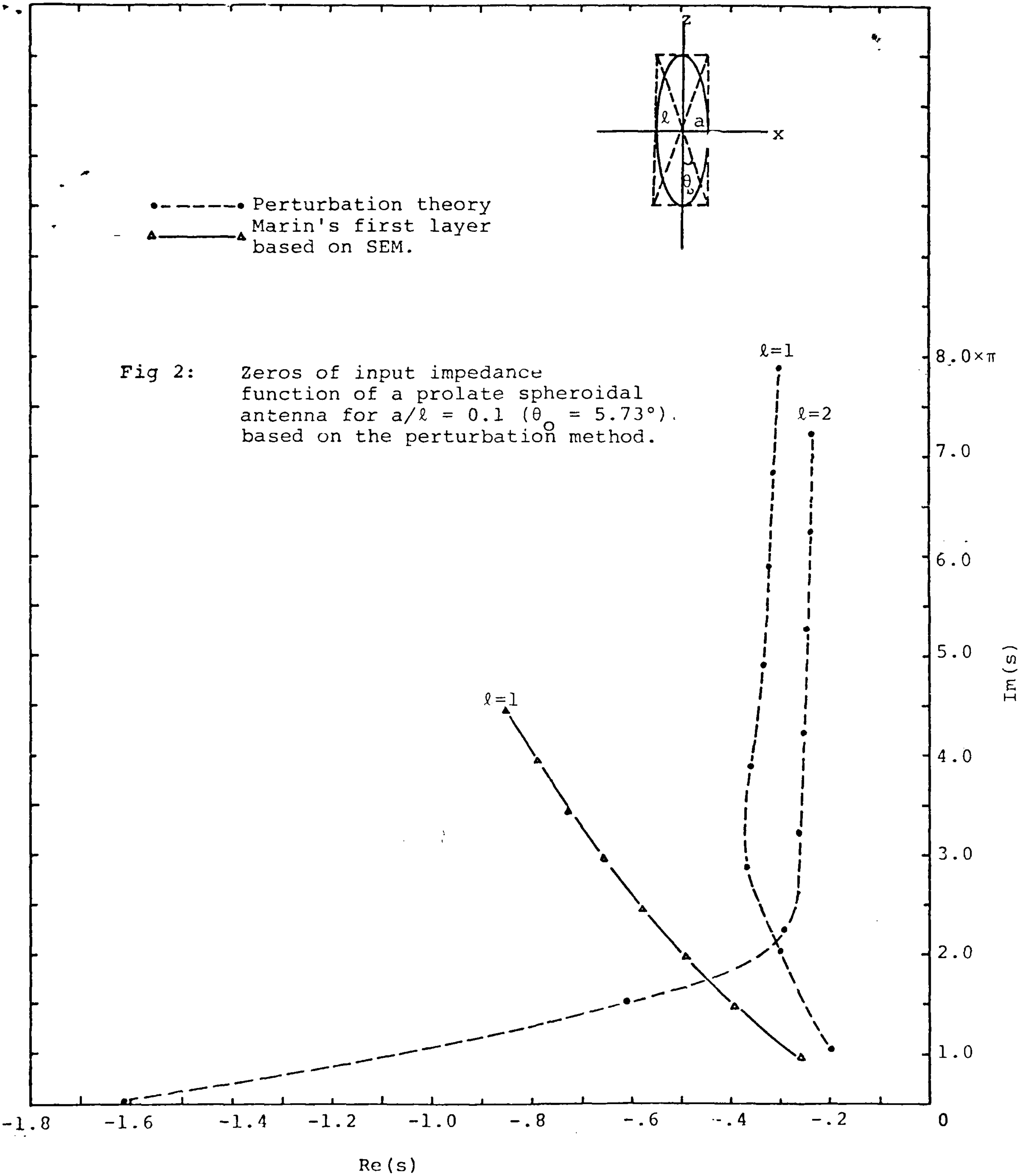

Fig 2: Zeros of input impedance function of a prolate spheroidal antenna for $a/\ell = 0.1$ ($\theta_o = 5.73°$) based on the perturbation method.

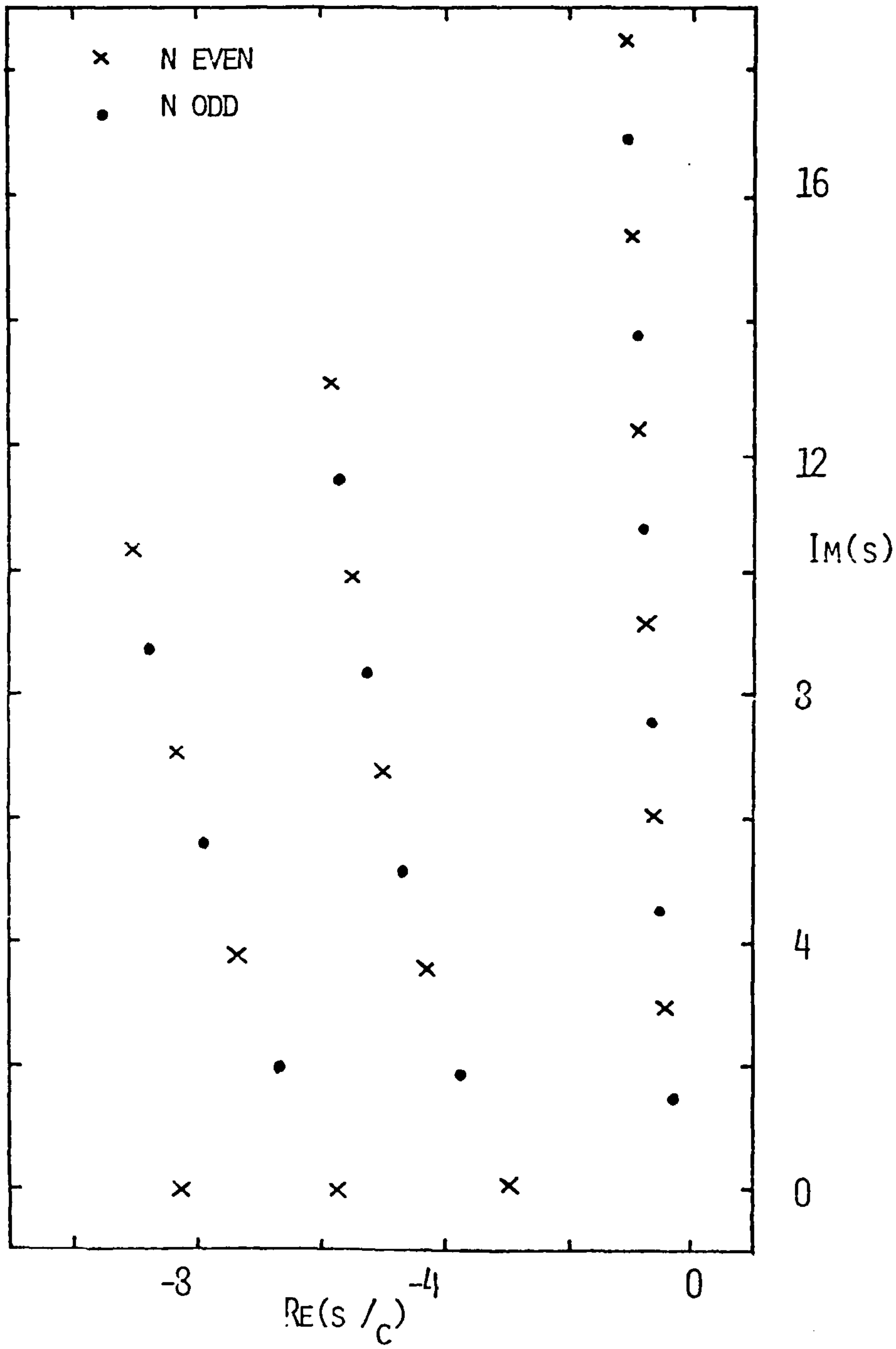

Fig. 2-1: POLES OF THE INDUCED CURRENT FUNCTION OF A PROLATE SPHEROIDAL SCATTERER OF a/ = 0.1 BASED ON THE SEM (AFTER L. MARIN)